LOCALLY MAXWELLIAN SOLUTIONS OF BOLTZMANN'S EQUATION

(LOKAL'NO-MAKSVELLOVSKIE RESHENIIA URAVNENIIA BOL'TSMANNA)

PMM Vol.29, № 5, 1965, pp. 973-976

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(Received February 18, 1965)

Boltzmann's kinetic equation (*)

$$\frac{\partial f}{\partial t} + \xi_i \frac{\partial f}{\partial x_i} + g_i \frac{\partial f}{\partial \xi_i} = J \tag{1}$$

is a complex integro-differential equation. At present, it is applicable only for the solutions of a very limited class of problems. Therefore, even highly degenerate solutions of this equation are of interest. The simplest exact solutions are those at which the collision integral J vanishes. These are the so-called locally Maxwellian solutions

$$f = \rho \left(\frac{m}{2\pi kT}\right)^{3/2} \exp\left(-\frac{mc^2}{2kT}\right) \qquad (\mathbf{c} = \boldsymbol{\xi} - \mathbf{u}) \tag{2}$$

Maxwell himself [1 to 3] has obtained stationary solutions of the type of (2), i.e. solutions satisfying the condition $\partial f/\partial t = 0$. In 1949 Grad [4] found all the locally Maxwellian solutions of Boltzmann's equation for the case where external forces are absent, i.e. for the condition $g_i = 0$. We shall show what conditions external fields must satisfy in order that solutions of Equation (1) having the form of (2) should exist, and also what solutions are possible for those fields.

Locally Maxwellian functions may be represented in a form somewhat different from (2),

$$\ln f = \gamma_0 + \gamma_i \xi_i + \gamma_4 \xi^2 \tag{3}$$

The connection between parameters in (3) and (2) is as follows:

$$2\frac{kT}{m} = -\frac{1}{\gamma_4}, \qquad u_i = -\frac{\gamma_i}{2\gamma_4}, \qquad \ln \rho \left(\frac{m}{2\pi kT}\right)^{3/2} = \gamma_0 - \frac{\gamma^2}{4\gamma_4} \tag{4}$$

Substituting (3) in (1), divided first by f , and equating to zero the coefficients of the various powers of ξ_i, we obtain the system of equations

$$\begin{gathered}\frac{\partial \gamma_0}{\partial t} + g_i \gamma_i = 0, \qquad \frac{\partial \gamma_i}{\partial t} + 2\gamma_4 g_i + \frac{\partial \gamma_0}{\partial x_i} = 0 \\ \frac{\partial \gamma_4}{\partial t}\delta_{ij} + \frac{1}{2}\left(\frac{\partial \gamma_i}{\partial x_j} + \frac{\partial \gamma_j}{\partial x_i}\right) = 0, \qquad \frac{\partial \gamma_4}{\partial x_i} = 0\end{gathered} \tag{5}$$

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*) Here and later a repeated index indicates summation.

First, we note that, substituting into this Equations (4), it is easy to show the equivalence of this system of equations with Euler's equations together with the condition that the stress tensor and temperature gradient are zero. In other words, solution of Euler's equations, which are, at the same time, solutions of the Navier-Stokes equations, define a locally Maxwellian distribution function which satisfies Boltzmann's equation.

Let us investigate the solutions of the system of Equations (5). Differentiating the third equation of this system with respect to x_k, then permuting indices cyclically, and adding together the resulting three equations, we find

$$\frac{\partial^2 \gamma_i}{\partial x_j \partial x_k} = 0$$

It follows that

$$\gamma_i(x, t) = a_i(t) + b_{ij}(t) x_j \tag{6}$$

Putting this into (5), we find that

$$b_{ij}(t) = \begin{cases} -b_{ji} & \text{for } i \neq j \\ -\partial\gamma_4/\partial t & \text{for } i = j \end{cases}$$

That is, Equation (6) may be written in the form (*)

$$\gamma(\mathbf{x}, t) = \mathbf{a}(t) - \dot{\gamma}_4(t)\mathbf{x} + [\omega(t) \times \mathbf{x}] \tag{7}$$

Equation (7) means that the motion of the gas is a superposition of three motions: solid body-like rotation, radial expansion, and translational motion.

We now simplify the equations defining γ_0 and $\mathbf{g}$. We write the first two equations of system (5) in vector form

$$\frac{\partial\gamma_0}{\partial t} + (\mathbf{g}, \gamma) = 0, \qquad \frac{\partial\gamma}{\partial t} + 2\gamma_4\mathbf{g} + \operatorname{grad}\gamma_0 = 0 \tag{8}$$

Applying the rot operator to the last equation, substituting into it expression (7) and assuming temperature to be finite, we obtain

$$\operatorname{rot}\mathbf{g}(\mathbf{x}, t) = -\frac{\dot{\omega}(t)}{\gamma_4(t)} \quad \text{or} \quad \mathbf{g}(\mathbf{x}, t) = \operatorname{grad}\Psi(\mathbf{x}, t) - \frac{1}{2}\left[\frac{\dot{\omega}(t)}{\gamma_4(t)} \times \mathbf{x}\right] \tag{9}$$

Putting (9) into the second equation of system (8), we obtain

$$\operatorname{grad}(2\gamma_4(t)\Psi(\mathbf{x}, t) + \gamma_0(\mathbf{x}, t)) = \ddot{\gamma}_4(t)\,\mathbf{x} - \dot{\mathbf{a}}(t)$$

and, integrating, find

$$\gamma_0(\mathbf{x}, t) = -2\gamma_4(t)\Psi(\mathbf{x}, t) + 0.5\ddot{\gamma}_4(t)x^2 - (\dot{\mathbf{a}}(t), \mathbf{x}) + b(t) \tag{10}$$

where $b(t)$ is an arbitrary function. Putting (10) in the first equation of (8), we have

$$-2\gamma_4\frac{\partial\Psi}{\partial t} + (\operatorname{grad}\Psi, \gamma) = 2\dot{\gamma}_4\Psi - 0.5\dddot{\gamma}_4 x^2 + (\ddot{\mathbf{a}}, \mathbf{x}) - \dot{b}(t) + 0.5\left(\left[\frac{\dot{\omega}}{\gamma_4} \times \mathbf{x}\right], \gamma\right) \tag{11}$$

Here γ is defined by Expression (7).

Thus, the density, velocity, and temperature are determined by Expressions (4),(7),(10) and the last equation of system (5), while the form of the potential consistent with a locally Maxwellian flow and the connection between the potential and the parameters defining ρ, $\mathbf{u}$ and T are found from Equation (11). Equation (11) is linear in first order partial derivatives. Its general solution is the sum of a particular solution of the nonhomogeneous equation and the general solution of the homogeneous equation $\Psi = \Psi_1 + \Psi_2$. Here, Ψ_1, the particular solution of the nonhomogeneous equation (11), has the form

$$\Psi_1(\mathbf{x}, t) = p_{ij}(t)x_i x_j + g_i(t)x_i + r(t) \tag{12}$$

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*) A dot over a letter means, as usual, differentiation with respect to time.

The connection between p_{ij} , g_i , r and the parameters of the motion, ω , $\mathbf{a}$, γ_4, b are found by substituting (12) into (11) and equating to zero the coefficients of the different powers of x_i. We note that, of the ten coefficients defining Ψ_1, only eight are independent; Ψ_2 is the general solution of Equation

$$-2\gamma_4 \frac{\partial \Psi_2}{\partial t} + (\text{grad}\, \Psi_2, \boldsymbol{\gamma}) = 2\gamma_4^{\cdot} \Psi_2 \tag{13}$$

From the theory of first-order equations it follows that Ψ_2 is determined from Equation $V(c_1, c_2, c_3, c_4) = 0$, where V is an arbitrary function and c_1, c_2, c_3, c_4 are first integrals of the characteristic system of Equations (13),

$$\frac{dx_1}{\gamma_1} = \frac{dx_2}{\gamma_2} = \frac{dx_3}{\gamma_3} = \frac{dt}{-2\gamma_4} = \frac{d\Psi_2}{2\gamma_4^{\cdot} \Psi_2} \tag{14}$$

Consequently, Ψ_2 has the form $\Psi_2 = \gamma_4^{-1}\Psi_3\ (c_5, c_6, c_7)$, where Ψ_3 is an arbitrary function of first integrals of system (14), not including the last equation. This means that Ψ_3 is conserved on certain lines in the four-dimensional space $(\mathbf{x}, t)$; these turn out to be the trajectories of points having the velocities $(\boldsymbol{\gamma}, -2\gamma_4)$ in that space.

Thus, in the case of external forces which depend on the coordinates and time, a locally Maxwellian flow may exist only if the forces are composed of two terms: a potential one and a rotational one. The second term has the form $[\boldsymbol{\sigma}(t) \times \mathbf{x}]$, where the function $\boldsymbol{\sigma}(t)$ characterizes the connection between the temperature and angular acceleration. The potential, however, is not an arbitrary function of the coordinates and time, but is determined in an arbitrary way by only three independent parameters. These parameters are found from system (13). The parameters of the motion, as has been stated, are determined from Equation (11).

For example, we shall consider the case of radical expansion, that is, the case where $\mathbf{a} = \boldsymbol{\omega} = \text{rot}\, \mathbf{g}(\mathbf{x}, t) = 0$. Then $\boldsymbol{\gamma} = -\gamma_4^{\cdot}\mathbf{x}$, and system (14) becomes

$$\frac{dx_1}{-\gamma_4^{\cdot} x_1} = \frac{dx_2}{-\gamma_4^{\cdot} x_2} = \frac{dx_3}{-\gamma_4^{\cdot} x_3} = \frac{dt}{-2\gamma_4} = \frac{d\Psi_2}{2\gamma_4^{\cdot}\Psi_2}$$

$$\Psi_2 = \frac{1}{\gamma_4(t)}\, \Psi_3 \left(\frac{x^2}{\gamma_4(t)}, \frac{x_1}{x_2}, \frac{x_1}{x_3}\right)$$

where Ψ_3 is an arbitrary function. The function Ψ_1 in this example is

$$\Psi_1 = \frac{\gamma_4^{\cdot\cdot\cdot}}{8\gamma_4^{\cdot} + 4\gamma_4}\, x^2$$

This means that only the potential, having the form

$$\Psi(\mathbf{x}, t) = f(t)\, x^2 + \frac{1}{\varphi(t)}\, \Phi\left(\frac{x^2}{\varphi(t)}, \frac{x_1}{x_2}, \frac{x_1}{x_3}\right)$$

corresponds to a locally Maxwellian solution of Boltzmann's equation; where φ is a negative and Φ an arbitrary function, while f is determined as

$$f(t) = \frac{\varphi^{\cdot\cdot\cdot}}{8\varphi^{\cdot} + 4\varphi}$$

Thus, we see that the potential in this case may be given as an arbitrary function of only three parameters: two angular parameters, and either a time or a radius. If we specify Ψ as a function of time, then the dependence on radius is also determined and, conversely, specifying Ψ as a function of the coordinates we determine the variation of the potential with time. For a given potential, we have the following expressions for the temperature, velocity, and density:

$$2\frac{kT}{m} = -\frac{1}{\varphi(t)}, \qquad \mathbf{u} = -\frac{\varphi^{\cdot}(t)}{2\varphi(t)}\,\mathbf{x}$$

$$\rho(\mathbf{x}, t) = \left(-\frac{\pi}{\varphi(t)}\right)^{3/2} \exp\left[-2\varphi(t)\,\Psi(\mathbf{x}, t) + 0.5\varphi^{\cdot\cdot}(t)\,x^2 - \frac{\varphi^{\cdot 2}(t)\,x^2}{4\varphi(t)}\right]$$

Up to now, we have been considering the case where the internal forces are specified as functions of time and radius. Before considering a more restricted class of solutions for $\mathbf{g} = \mathbf{g}(\mathbf{x})$, we note that the case $\Psi = \Psi(t)$ is entirely equivalent to the case $\Psi = 0$. In fact, from Equation (11) we see at once that $\boldsymbol{\omega}^{\cdot} = 0$, i.e. $\mathbf{g} = 0$, and, therefore, the parameters of the motion do not depend on the potential.

We shall now consider in more detail the case $\mathbf{g} = \mathbf{g}(\mathbf{x})$ and show that for this case the problem of finding the form of the internal forces and the flows determined by those forces can be solved completely. Since $\mathbf{g} = \mathbf{g}(\mathbf{x})$, Equation (9) takes the form

$$\mathbf{g}(\mathbf{x}) = \operatorname{grad} \Psi(\mathbf{x}) - 0.5\,[\boldsymbol{\Omega} \times \mathbf{x}] \qquad (\boldsymbol{\Omega} \text{ is a constant vector})$$

Therefore,

$$\boldsymbol{\omega}(t) = \Gamma(t)\boldsymbol{\Omega} + \boldsymbol{\omega}_0 \qquad (\Gamma(t) \text{ is the initial of } \gamma_4(t))$$

Equation (11) takes the form

$$(\operatorname{grad} \Psi, \gamma) = 2\gamma_4^{\cdot}\,\Psi - 0.5\gamma_4^{\cdot\cdot\cdot}x^2 + (\mathbf{a}^{\cdot\cdot}, \mathbf{x}) - b^{\cdot} + 0.5\Gamma(t)\,|[\boldsymbol{\Omega} \times \mathbf{x}]|^2 + 0.5\,([\boldsymbol{\Omega} \times \mathbf{x}], \mathbf{a} + [\boldsymbol{\omega}_0, \mathbf{x}]) \tag{15}$$

As already stated, the case $\Psi = \text{const}$ is entirely equivalent to the case $\Psi = 0$. Moreover,

$$\mathbf{a}^{\cdot\cdot} = \boldsymbol{\Omega} = 0, \qquad \gamma_4^{\cdot\cdot} = 0$$

i.e. we obtain Grad's solution [4]. Let $\Psi \not\equiv 0$ and $\boldsymbol{\omega}_0 \neq 0$; then (since $\Gamma^{\cdot} = \gamma_4 \neq 0$)

$$\boldsymbol{\Omega} = \mathbf{a}^{\cdot} = 0, \qquad \gamma_4^{\cdot\cdot} = b^{\cdot\cdot} = 0$$

and we have

$$(\operatorname{grad} \Psi, \gamma) = 2\gamma_4^{\cdot}\Psi, \qquad \gamma = \mathbf{a} - \gamma_4\mathbf{x} + [\boldsymbol{\omega}_0 \times \mathbf{x}] = \mathbf{a} + A\mathbf{x} \tag{16}$$

Here, A is a constant matrix. The function Ψ is determined from the characteristic system

$$\frac{dx_1}{\gamma_1} = \frac{dx_2}{\gamma_2} = \frac{dx_3}{\gamma_3} = \frac{d\Psi}{2\gamma_4^{\cdot}\Psi} \tag{17}$$

Going over to a new variable $\mathbf{y} = \mathbf{x} + A^{-1}\mathbf{a}$, and then to a system of coordinates in which one of the axes is along the direction of the constant vector $\boldsymbol{\omega}_0$, we reduce system (17) to

$$\frac{dz_i}{\gamma_i'} = \frac{dz_1}{-\gamma_4^{\cdot}z_1 - \omega_0 z_2} = \frac{dz_2}{-\gamma_4^{\cdot}z_2 + \omega_0 z_1} = \frac{dz_3}{-\gamma_4^{\cdot}z_3} = \frac{d\Psi}{2\gamma_4^{\cdot}\Psi} \tag{18}$$

Then

$$\Psi = z_3^{-2}\Psi_3 \tag{19}$$

where Ψ_3 satisfies Equation $(\operatorname{grad}\Psi_3, \gamma') = 0$, that is, Ψ_3 is constant along the trajectory of a point moving with the velocity γ'. In other words, Ψ_3 is an arbitrary function of the first integrals of the first two equations of system (18). These integrals are

$$\frac{z_3^2}{z_1^2 + z_2^2} = c_1, \qquad \frac{\gamma_4^{\cdot}}{\omega_0}\tan^{-1}\frac{1}{2}\left(\frac{z_2}{z_1} - \frac{z_1}{z_2}\right) - \ln|z_1^2 + z_2^2| = c_2$$

Thus, in this case $(\Psi(\mathbf{x}) \not\equiv \text{const}, \ \boldsymbol{\omega}_0 \neq 0)$ γ is constant, and γ_4 and γ_0 change linearly with time. The dependence of the coordinates is given by Expressions (16) and (10). If we set $\gamma_4^{\cdot} = 0$, we obtain Maxwell's solutions [1 and 2].

If $\boldsymbol{\omega}_0 = 0$, then all the coefficients in Equation (15) which depend on time must be proportional to each other,

$$\gamma_4^{\cdot\cdot\cdot}(t) = -\frac{\gamma_4^{\cdot}(t)}{\gamma} = \frac{b^{\cdot}(t)}{\beta} = \frac{a_i(t)}{\alpha_i}$$

(from the condition of the finitude of mass we obtain $\gamma > 0$). Then the potential is determined as $\Psi = \Psi_1 + \Psi_2$, where

$$\Psi_1 = p_{ij}x_ix_j + q_ix_i + r$$

(now p_{ij}, q_i and r are constant), and Ψ_2 is determined from Equation

$$(\operatorname{grad} \Psi_2, \boldsymbol{\alpha} + \gamma \mathbf{x} + \gamma^2 [\boldsymbol{\Omega} \times \mathbf{x}]) = -2\gamma\Psi_2$$

In this case we find the function Ψ_2 in the same way as for the case $\omega_0 \neq 0$.

Thus, for $\omega_0 = 0$, that is, in the absence of a constant component of rotation, oscillatory solutions are possible for γ_4, γ and γ_0. The density, velocity and temperature are found, as before, from the Expressions (4).

To sum up the above, we state that for internal force fields independent of time, their form can be determined and the flows corresponding to those fields found. The solution may be devided into two classes: for constant rotation of the gas, the parameters γ_4 and γ_0 are linear in time; in the absence of a constant component of angular velocity, oscillatory solutions for γ, γ_4 and γ_0 are possible.

The calculations can be significantly simplified if one considers separately translational motion, radial expansion, or solid body-like rotation of the gas.

In conclusion, the author thanks M.N.Kogan for suggesting the problem and for discussions.

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Translated by A.R.